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Lacunary statistical convergence of Bernstein operator sequences

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ABSTRACT

The Bernstein operator is one of the important topics of approximation theory in which it has been studied in great details for a long time. Recently the statistical convergence of Bernstein operators was studied. In this paper, by using the concept of natural density and lacunary sequences we first introduce the notion of lacunary statistical convergence of a sequence of Bernstein polynomials. Next we apply this notion to V_B^θ-summability. We also investigate some inclusion relations related to these concepts.

1. Introduction

The idea of statistical convergence for single sequences was introduced by Fast (1951) and then studied by various authors, e.g., Šalát (1980), Fridy (1985), Connor (1988), Esi (2006) and many others. This notion was used by Kolk (1988) to extend statistical convergence to normed spaces and Maddox (1988) extended to locally convex spaces. Cakalli (1995) extended lacunary statistical convergence to topological groups.

Let $K \subseteq \mathbb{N}$. Then $\delta(K) = \lim_n \frac{1}{n}|\{k \leq n : k \in K\}|$ is said to be natural density of the set K.

As known, the Bernstein operator of order n is given by:

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k} \qquad (1)$$

where f is a continuous (real or complex valued) function defined on $[0,1]$. $B_n(f;x)$ was introduced in 1912 in Bernstein's constructive proof of the Weierstrass approximation theorem cf (Acikgoz and Araci, 2010; Lorentz, 1986; Oruc and Phillips, 1999).

We now give Bernstein's theorem.

Theorem 1. *Given a function $f \in C[0,1]$ and any $\varepsilon > 0$, there exists an integer N such that*:

$$|f(x) - B_n(f;x)| < \varepsilon$$

for all $n \geq N$ and $0 \leq x \leq 1$.

By Theorem 1, we note that if f is merely bounded on $[0,1]$, the sequence $\left(B_n(f,x)\right)_{n=1}^{\infty}$ converges to $f(x)$ at any point in which f is continuous. As a remarkable property, we note further that the derivatives of Bernstein operator converge to the derivatives of the function cf. (Lorentz, 1986).

By a lacunary sequence $\theta = (k_r)$; $r = 0,1,2,3,\ldots$ where $k_0 = 0$, we shall mean an increasing sequence of nonnegative integers with $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$. the ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r.

Let f be a continuous function defined on the closed interval $[0,1]$. A sequence of Bernstein polynomials $\left(B_n(f,x)\right)$ is said to be statistically convergent or s_B-convergent to f if for every $\varepsilon > 0$ the set $K_\varepsilon := \{k \in \mathbb{N} : |B_n(f,x) - f(x)|\}$ has natural density zero, i.e., $\delta(K_\varepsilon) = 0$. That is:

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : |B_k(f,x) - f(x)| \geq \varepsilon\}| = 0.$$

In this case, we write as δ-lim$B_n(f,x) = f(x)$ or $B_n(f,x)\, \underline{S_B}\, f(x)$ (Esi et al., 2016).

Let f be a continuous function defined on the closed interval $[0,1]$. Then a sequence of Bernstein polynomials $\left(B_n(f,x)\right)$ is said to be strongly Cesàro summable of V_B-summable to $f(x)$ if (Esi et al., 2016):

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} |B_k(f,x) - f(x)| = 0.$$

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In the next part, we give some definitions and notations which will be useful in deriving the main results of this paper.

2. Definitions and notations

The aim of this study is to study lacunary statistical convergence and lacunary strongly summable of sequences of Bernstein polynomials.

Definition 1. Let $\theta = (k_r)$ be lacunary sequence and f be a continuous function defined on the closed interval $[0,1]$. A sequence of Bernstein polynomials $(B_n(f,x))$ is said to be lacunary statistically convergent or S_B^θ-convergent to f if for every $\varepsilon > 0$ the set $K_\varepsilon := \{k \in I_r : |B_n(f,x) - f(x)|\}$ has natural density zero, i.e., $\delta^\theta(K_\varepsilon) = 0$. That is

$$\lim_r \frac{1}{h_r} |\{k \in I_r : |B_k(f,x) - f(x)| \geq \varepsilon\}| = 0.$$

Furthermore, let S_B^θ denotes the set of lacunary convergent sequence of Bernstein polynomials $(B_n(f,x))$.

Definition 2. Let $\theta = (k_r)$ be lacunary sequence and f be a continuous function defined on the closed interval $[0,1]$. A sequence of Bernstein polynomials $(B_n(f,x))$ is said to be lacunary strongly summable or V_B^θ*-summable* to f if*:*

$$\lim_r \frac{1}{h_r} \sum_{k \in I_r} |B_k(f,x) - f(x)| = 0.$$

Furthermore, let V_B^θ denotes the set of lacunary strongly summable sequence of Bernstein polynomials $(B_n(f,x))$.

3. Main results

Theorem 2. Let $\theta = (k_r)$ be lacunary sequence. Then,

(a) If $B_n(f,x) \to f(x)(V_B^\theta)$ then $B_n(f,x) \to f(x)(S_B^\theta)$,

(b) If Bernstein polynomials sequence $(B_n(f,x))$ is bounded and $B_n(f,x) \to f(x)(S_B^\theta)$ then $B_n(f,x) \to f(x)(V_B^\theta)$,

(c) $S_B^\theta \cap \ell_\infty(B) = V_B^\theta \cap \ell_\infty(B)$

where $\ell_\infty(B)$ the set of all bounded sequences of Bernstein polynomials.

Proof. (a): If $\varepsilon > 0$ and $B_n(f,x) \to f(x)(V_B^\theta)$, then

$$\sum_{k \in I_r} |B_k(f,x) - f(x)| \geq \sum_{k \in I_r | B_n(f,x) - f(x)| \geq \varepsilon} |B_k(f,x) - f(x)| \geq \varepsilon . |\{k \in I_r : |B_k(f,x) - f(x)| \geq \varepsilon\}|.$$

Therefore $B_n(f,x) \to f(x)(S_B^\theta)$.

(b): Suppose that $(B_n(f,x))$ is bounded and $B_n(f,x) \to f(x)(S_B^\theta)$. Then we can assume that $|B_k(f,x) - f(x)| \leq T$ for all k. Given $\varepsilon > 0$;

$$\frac{1}{h_r} \sum_{k \in I_r} |B_k(f,x) - f(x)| = \frac{1}{h_r} \sum_{k \in I_r | B_k(f,x) - f(x)| \geq \varepsilon} |B_k(f,x) - f(x)| + \frac{1}{h_r} \sum_{k \in I_r | B_k(f,x) - f(x)| < \varepsilon} |B_k(f,x) - f(x)| \leq \frac{T}{h_r} |\{k \in I_r : |B_k(f,x) - f(x)| \geq \varepsilon\}| + \varepsilon.$$

Therefore $B_n(f,x) \to f(x)(V_B^\theta)$.

(c): It follows from (a) and (b).

Theorem 3. Let $\theta = (k_r)$ be lacunary sequence with $liminf_r q_r > 1$, then $B_n(f,x)\ \underline{S_B}\ f(x)$ implies $B_n(f,x)\ \underline{S_B^\theta}\ f(x)$, where S_B (Esi et al., 2016) the set of all $(B_n(f,x))$ of Bernstein polynomials sequence such that;

$$\lim_n \frac{1}{n} |\{k \leq n : |B_k(f,x) - f(x)| \geq \varepsilon\}| = 0.$$

Proof. Suppose that $liminf_r q_r > 1$, then there exists a $\delta > 0$ such that $q_r \geq 1 + \delta$ for sufficiently large r, which implies

$$\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta}$$

if $B_n(f,x)\ \underline{S_B}\ f(x)$, then for every $\varepsilon > 0$ and for sufficiently large r, we have

$$\frac{1}{k_r} |\{k \leq k_r : |B_k(f,x) - f(x)| \geq \varepsilon\}| \geq \frac{1}{k_r} |\{k \in I_r : |B_k(f,x) - f(x)| \geq \varepsilon\}| \geq \frac{\delta}{1+\delta} . \frac{1}{h_r} |\{k \in I_r : |B_k(f,x) - f(x)| \geq \varepsilon\}|.$$

This complete the proof.

Theorem 4 Let $\theta = (k_r)$ be lacunary sequence with $sup_r q_r < \infty$, then $B_n(f,x)\ \underline{S_B^\theta}\ f(x)$ implies $B_n(f,x)\ \underline{S_B}\ f(x)$.

Proof. If $\limsup_r q_r < \infty$,then there exists $B > 0$ such that $q_r < C$ for all $r \geq 1$. Let $X \overset{S_\theta^L(F)}{\sim} Y$ and $\varepsilon > 0$. There exists $B > 0$ such that for every $j \geq B$

$$A_J = \frac{1}{h_j} \left|\{k \in I_j : |B_k(f,x) - f(x)| \geq \varepsilon\}\right| < \varepsilon.$$

We can also find $K > 0$ such that $A_j < K$ for all $j = 1,2,3, \ldots$. Now let n be any integer with $k_{r-1} < n < k_r$, where $r \geq B$. Then:

$$\begin{aligned}
&\frac{1}{n} |\{k \leq n : |B_k(f,x) - f(x)| \geq \varepsilon\}| \\
&\leq \frac{1}{k_{r-1}} |\{k \leq k_r : |B_k(f,x) - f(x)| \geq \varepsilon\}| \\
&= \frac{1}{k_{r-1}} |\{k \in I_1 : |B_k(f,x) - f(x)| \geq \varepsilon\}| + \frac{1}{k_{r-1}} |\{k \in I_2 : |B_k(f,x) - f(x)| \geq \varepsilon\}| \\
&+ \cdots + \frac{1}{k_{r-1}} |\{k \in I_r : |B_k(f,x) - f(x)| \geq \varepsilon\}| \\
&= \frac{k_1}{k_{r-1} k_1} |\{k \in I_1 : |B_n(f,x) - f(x)| \geq \varepsilon\}| \\
&+ \frac{k_2 - k_1}{k_{r-1}(k_2 - k_1)} |\{k \in I_2 : |B_k(f,x) - f(x)| \geq \varepsilon\}| \\
&+ \cdots + \frac{k_B - k_{B-1}}{k_{r-1}(k_B - k_{B-1})} |\{k \in I_B : |B_k(f,x) - f(x)| \geq \varepsilon\}|
\end{aligned}$$

$+\cdots+\frac{k_r-k_{r-1}}{k_{r-1}(k_r-k_{r-1})}|\{k\in I_r: |B_k(f,x)-f(x)|\geq\varepsilon\}|$
$=\frac{k_1}{k_{r-1}}A_1+\frac{k_2-k_1}{k_{r-1}}A_2+\cdots+\frac{k_B-k_{B-1}}{k_{r-1}}A_B+\cdots+\frac{k_r-k_{r-1}}{k_{r-1}}A_r$
$\leq\left\{\sup_{j\geq1}A_j\right\}\frac{k_B}{k_{r-1}}+\left\{\sup_{j\pm B}A_j\right\}\frac{k_r-k_B}{k_{r-1}}$
$\leq K.\frac{k_B}{k_{r-1}}+\varepsilon.C.$

This complete the proof.

Theorem 5. Let $\theta=(k_r)$ be lacunary sequence $1<liminf_r q_r\leq limsup_r q_r<\infty$, then $B_n(f,x)\ \underline{S_B}\ f(x)\Leftrightarrow B_n(f,x)\ \underline{S_B^\theta}\ f(x)$.

Proof. The result follow from Theorem 3 and Theorem 4.

We now consider the inclusion of $S_B^{\theta^i}$ by S_B^θ, where θ^i is lacunary refinement of θ. Recall [9] that the lacunary sequence $\theta^i=(k_r^i)$ is called lacunary refinement of the lacunary sequence $\theta=(k_r)$ if $(k_r)\subset(k_r^i)$.

Theorem 6. *If θ^i is a lacunary refinement of θ and $B_n(f,x)\ \underline{S_B^{\theta^i}}\ f(x)$, then $B_n(f,x)\ \underline{S_B^\theta}\ f(x)$.*

Proof. Suppose that each I_r of θ contains the points $\left(k_{r,i}^i\right)_{i=1}^{v(r)}$ of θ^i so that;

$k_{r-1}<k_{r,1}^i<k_{r,2}^i<\cdots<k_{r,v(r)}^i=k_r$, where $I_{r,i}^i=(k_{r,i-1}^i,k_{r,i}^i]$.

Note that for all $r, v(r)\geq1$ because $(k_r)\subset(k_r^i)$. Let $\left(I_j^*\right)_{j\geq1}$ be the sequence of abutting intervals $\left(I_{r,i}^i\right)$ ordered by increasing right end points. Since $B_n(f,x)\ \underline{S_B^{\theta^i}}\ f(x)$, we get for each $\varepsilon>0$,

$$\lim_j\sum_{I_j^*\subset I_r}\frac{1}{h_j^*}\left|\left\{k\in I_j^*: |B_k(f,x)-f(x)|\geq\varepsilon\right\}\right|=0. \qquad (2)$$

As before we write, $h_r=k_r-k_{r-1}$, $h_{r,i}^i=k_{r,i}^i-k_{r,i-1}^i$ and $h_{r,1}^i=k_{r,1}^i-k_{r-1}$. For each $\varepsilon>0$, we have

$$\frac{1}{h_r}|\{k\in I_r: |B_k(f,x)-f(x)|\geq\varepsilon\}|= \qquad (3)$$
$$\frac{1}{h_r}\sum_{I_j^*\subset I_r}h_j^*\frac{1}{h_j^*}\left|\left\{k\in I_j^*: |B_k(f,x)-f(x)|\geq\varepsilon\right\}\right|=$$
$$\frac{1}{h_r}\sum_{I_j^*\subset I_r}h_j^*\left(V_{\chi_k}^{\theta^i}\right)_j \qquad (4)$$

where χ_k is the characteristic function of the set $K=\left\{k\in I_j^*: |B_k(f,x)-f(x)|\geq\varepsilon\right\}$ and $V_{\chi_k}^{\theta^i}$, where;

$$V_{\chi_k}^{\theta^i}=\begin{cases}\frac{1}{h_j^*}, \text{if} k\in I_j^*\\ 0, \text{if} k\notin I_j^*\end{cases}.$$

By (Eq. 2), $\left(V_{\chi_k}^{\theta^i}\right)$ is a null sequence and the equality (Eq. 3) is a regular weighted mean transform of $V_{\chi_k}^{\theta^i}$. Hence, the transform (Eq. 3) also to goes zero as $r\to\infty$. This completes the proof.

4. Conclusion

In recent years the statistical convergence has been adapted to the sequences of fuzzy numbers, interval numbers, etc. In the paper, we have applied lacunary statistical convergence to Bernstein polynomials defined in Eq. 1. From this point of view, we have introduced some new definitions and proved some theorems of the introduced sequence spaces.

In our next works, we will try to apply this idea to other convergence types.

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